UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

NON-LINEAR DIFFERENTIAL EQUATIONS HAVING BOTH CUBIC DAMPING AND STIFFNESS

by

A.W. Babister, M.A., Ph.D.

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SUMMARY

The nature of solutions of the autonomous equation

$$\ddot{x} + b_0\dot{x} + \nu\,\dot{x}^3 + c_1 x + c_3 x^3 = 0$$

is considered. Trajectories in the $(x,\dot{x})$ phase plane are given for various combinations of signs of b_0, ν, c_1 and c_3. It is well known that limit cycles can occur if both c_1 and c_3 are positive. Limit cycles are shown to occur if c_1 and c_3 have opposite signs, and their stability is investigated.

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General Introduction

This report is one of a series dealing with dynamical systems with one degree of freedom which satisfy non-linear differential equations of the second order (see Babister, 1973, 1974 and 1975). In a previous report we considered the nature of solutions of differential equations of the form

$$\frac{d^2x}{dt^2} + (b_0 + b_2x^2)\frac{dx}{dt} + c_1x + c_3x^3 = 0. \qquad (1)$$

Here we shall consider solutions of the equation

$$\frac{d^2x}{dt^2} + b_0\frac{dx}{dt} + \nu\left(\frac{dx}{dt}\right)^3 + c_1x + c_3x^3 = 0. \qquad (2)$$

or of the equivalent plane autonomous system

$$\left.\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= -b_0y - \nu y^3 - c_1x - c_3x^3 ,\end{aligned}\right\} \qquad (3)$$

where b_0, c_1, c_3 and ν are real constants. These equations occur in the analysis of the oscillatory response of large amplitude motions of aircraft and rockets.

Here x can be thought of as the displacement at time t. We shall consider both the variation of x with t and also the trajectories in the (x,y) phase plane. Every solution of (3) is uniquely determined by the initial conditions $x = x_0$, $y = \dot{x}_0 = y_0$. We shall be particularly interested in those cases in which periodic solutions or limit cycles occur. If $c_3 = 0$ and b_0 and ν have opposite signs, equation (2) reduces to Rayleigh's equation, which is an example of an autonomous system that possesses a limit cycle. We shall show that limit cycles are also possible if $c_3 \neq 0$. Any periodic solution of (2) is represented by a closed curve in the phase plane, and this curve intersects the x axis twice (at points corresponding to the maximum and minimum values of x on the closed curve).

For the system considered, the point $x = 0$ is an equilibrium position. In the (x,y) phase plane, the origin is a singular point. By considering the signs of the coefficients in (2), we find that this point is a focus (or node) if $c_1 > 0$ and a saddle point if $c_1 < 0$. The focus at 0 will be stable or unstable according as $b_0 >$ or < 0. As shown by Loud (1964), the focus becomes a centre if $b_0 = \nu = 0$ (with $c_1 > 0$).

If both c_1 and c_3 have the same sign, 0 is the only singular point. If $c_1 c_3 < 0$, (3) has two other singular points at $x = \pm\sqrt{-c_1/c_3}$. If we write $x = z + \gamma$, (2) becomes

$$\frac{d^2z}{dt^2} + b_0 \frac{dz}{dt} + \nu\left(\frac{dz}{dt}\right)^3 + c_1\gamma + c_3\gamma^3 + (c_1+3c_3\gamma^2)z + 3c_3\gamma z^2 + c_3 z^3 = 0. \quad (4)$$

If $\gamma^2 = -c_1/c_3$, (4) reduces to

$$\frac{d^2z}{dt^2} + b_0\frac{dz}{dt} + \nu\left(\frac{dz}{dt}\right)^3 - 2c_1 z + 3c_3\gamma z^2 + c_3 z^3 = 0. \quad (5)$$

Thus the points $x = \pm\sqrt{-c_1/c_3}$ will both be foci (or nodes) if $c_1 < 0$ and will both be saddle points if $c_1 > 0$. These foci will be stable or unstable according as $b_0 >$ or < 0. More generally, from (4), any equation of the form

$$\frac{d^2z}{dt^2} + B_0\frac{dz}{dt} + B_2\left(\frac{dz}{dt}\right)^3 + C_0 + C_1 z + C_2 z^2 + C_3 z^3 = 0 \quad (6)$$

with $C_3 \neq 0$, can be put in the same form as (2) with real coefficients on letting $x = z + \gamma$ where $\gamma = C_2/3C_3$, $C_0 - C_1\gamma + C_2\gamma^2 - C_3\gamma^3 = 0$.

In (2), put $x = \alpha X$, $t = \beta T$, where α and β are constants. Then

$$\frac{d^2X}{dT^2} + \beta b_0\frac{dX}{dT} + \frac{\alpha^2\nu}{\beta}\left(\frac{dX}{dT}\right)^3 + \beta^2 c_1 X + \alpha^2\beta^2 c_3 X^3 = 0. \quad (7)$$

Thus, if (2) has the solution $x = \phi(t)$, with $x = x_0$, $y = y_0$ at $t = 0$,

(7) has the solution $X = \alpha^{-1}\phi(\beta T)$, with $X = x_0/\alpha$ and $dX/dT = \beta y_0/\alpha$ at $T = 0$. We note that a variation in the value of α merely affects the non-linear terms in (7). If α is replaced by $-\alpha$, (7) is unchanged but the initial values of X and dX/dT both change sign. It follows that, if $x(t)$, $y(t)$ is a solution of (3), so is $-x(t)$, $-y(t)$. In particular, if (3) has a periodic solution, represented by a closed curve encircling the origin and passing through the point (x,y) in the phase plane, it is readily shown that the point $(-x,-y)$ also lies on this curve. Again, if $\alpha = 1$ and $\beta = -1$, the coefficients b_0 and ν in (2) become $-b_0$ and $-\nu$, and the variation of X with T is identical with that of x with $(-t)$.

Scaling factors were used in the numerical solutions given in this report, many of which were calculated on Glasgow University's analogue computer (PACE). The computer calculations were carried out for the equation

$$0.01\frac{d^2X}{dT^2} + 0.01b_0\frac{dX}{dT} + 0.16\nu\left(\frac{dX}{dT}\right)^3 + 0.01c_1X + 0.16c_3X^3 = 0 \qquad (8)$$

with the numerical values of b_0, c_1, c_3 and ν less than or equal to 1. Thus the solutions were performed in real time ($\beta = 1$) with a scaling factor $\alpha = 4$. The magnitudes of the initial conditions were then never greater than unity.

In part 1 of this report we discuss the nature of solutions of the differential equation (2) with $c_3 = 0$, and in part 2 we deal with the case $c_3 \neq 0$. For ease of reference the various cases are set out in table 1.

TABLE 1

Index to discussion of solutions of

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1x + c_3x^3 = 0$$

Para.	Case	ν	c_1	c_3	General remarks
		PART ONE			
1.2	1	0	0	0	
1.2	2	+	0	0	
1.2	3	-	0	0	
1.3	4	0	+	0	Periodic solutions
1.3	5	+	+	0	Some limit cycles
1.3	6	-	+	0	
1.4	7	0	-	0	
1.4	8	+	-	0	
1.4	9	-	-	0	
		PART TWO			
2.2	10	0	0	+	Periodic solutions
2.2	11	+	0	+	Some limit cycles
2.2	12	-	0	+	
2.3	13	0	0	-	
2.3	14	+	0	-	
2.3	15	-	0	-	
2.4	16	0	+	+	Periodic solutions
2.4	17	+	+	+	Some limit cycles
2.4	18	-	+	+	
2.5	19	0	+	-	Periodic solutions
2.5	20	+	+	-	Some limit cycles
2.5	21	-	+	-	
2.6	22	0	-	+	Periodic solutions
2.6	23	+	-	+	Some limit cycles
2.6	24	-	-	+	
2.7	25	0	-	-	
2.7	26	+	-	-	
2.7	27	-	-	-	

PART 1

Non-linear differential equation of the form

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x = 0$$

1.1 Introduction

We consider solutions of the equation

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x = 0 \tag{9}$$

or of the equivalent system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -b_0 y - \nu y^3 - c_1 x , \end{aligned}\right\} \tag{10}$$

where b_0, c_1 and ν are real constants. The initial conditions are taken as $x = x_0$, $y = \dot{x}_0 = y_0$ at time $t = 0$.

1.2 Systems with zero stiffness ($c_1 = 0$)

Case 1 $\nu = 0,\ c_1 = 0$

Equation (9) is then a linear differential equation. This case was dealt with in a previous report (Babister, 1973).

Case 2 $\nu > 0,\ c_1 = 0$

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 = 0 \tag{11}$$

Equation (11) has the following forms of first integral:

$$b_0 > 0, \quad \dot{x} = -\sqrt{b_0/\nu}\ \tan\left[\sqrt{b_0\nu}\ (x-A)\right], \tag{12}$$

$$b_0 = 0, \quad \dot{x} = 1/\left[\nu\ (x-A)\right], \tag{13}$$

$$b_0 < 0, \quad \dot{x} = \sqrt{|b_0/\nu|}\ \tanh\left[\sqrt{|b_0\nu|}\ (x-A)\right], \quad \left(|\dot{x}_0| < \sqrt{|b_0/\nu|}\right), \tag{14}$$

$$b_0 < 0, \quad \dot{x} = \sqrt{|b_0/\nu|}\ \coth\left[\sqrt{|b_0\nu|}\ (x-A)\right], \quad \left(|\dot{x}_0| > \sqrt{|b_0/\nu|}\right). \tag{15}$$

In (12) - (15), A is a constant. We note that (11) also has the particular integrals $\dot{x} = 0$ and $\dot{x} = \pm\sqrt{-b_0/\nu}$. As shown below, these are separatrices in the (x,y) phase plane.

The complete solutions of (11) are given by:

$$b_0 > 0, \quad \nu > 0, \quad b_0(t+B) = -\log|\sin\sqrt{b_0\nu}\,(x-A)|\,, \tag{16}$$

$$b_0 = 0, \quad (t+B) = \tfrac{1}{2}\nu(X-A)^2\,, \tag{17}$$

$$b_0 < 0, \quad \nu > 0, \quad b_0(t+B) = -\log|\sinh\sqrt{|b_0\nu|}\,(x-A)|\,, \quad (|\dot{x}_0|<\sqrt{|b_0\nu|}) \tag{18}$$

$$b_0 < 0, \quad \nu > 0, \quad b_0(t+B) = -\log|\cosh\sqrt{|b_0\nu|}\,(x-A)|\,, \quad (|\dot{x}_0|>\sqrt{|b_0\nu|}) \tag{19}$$

In (16)-(19), B is a constant.

Trajectories in the phase plane are shown for $b_0 = 0$ (fig. 1), $b_0\nu = 1$ (fig.2) and $b_0\nu = -1$ (fig. 3), in which νy is plotted against x. If $y_0 = 0$, x is constant for all values of t ; the phase plane curve is then a point. Thus there is a whole line of equilibrium points along the x axis. As can be seen from eq. (16) and fig. 2, if $b_0 > 0$, $x \rightarrow A$ as $t \rightarrow \infty$; thus all trajectories converge to some stable equilibrium point on the x axis. If $b_0 \leqslant 0$, from (17)-(19), $x \rightarrow \pm\infty$ as $t \rightarrow \infty$ for all trajectories for which $y_0 \neq 0$, as shown in figs. 1 and 3 ; in that case the equilibrium positions on the x axis are unstable. We see that, if $b_0\nu < 0$, the particular integrals $y = \pm\sqrt{-b_0/\nu}$ correspond to two separatrices, all trajectories converging on these two lines as $t \rightarrow \infty$.

<u>Case 3</u> $\nu < 0$, $c_1 = 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 = 0. \tag{20}$$

On putting $t = -T$, (20) becomes

$$\frac{d^2x}{dT^2} - b_0\frac{dx}{dT} - \nu\left(\frac{dx}{dT}\right)^3 = 0 \tag{21}$$

which is of the same form as (11). Thus the integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 2. The trajectories in the phase plane can be determined from (12)-(15), allowance being made for the sign changes in b_0 and ν in (20) and (21). In the phase plane, the point $(x,\nu y)$ in figures 1-3 is transformed into the point $(x, -|\nu|y)$, for systems having the same value of $b_0\nu$; in addition the arrows on the curves should be reversed, Thus, in this case, the solutions of (20) are unbounded and divergent as $t \to \infty$, except that, if $b_0\nu < 0$ and $|y_0| < \sqrt{-b_0/\nu}$, trajectories converge on stable equilibrium points $(x = A)$ on the x axis.

1.3 Systems with damping and positive stiffness

Case 4 $\nu = 0$, $c_1 > 0$.

Equation (9) is then a linear differential equation. This case was dealt with in a previous report (Babister, 1973).

Case 5 $\nu > 0$, $c_1 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x = 0. \tag{22}$$

This is an equation of the form

$$\ddot{x} + f(\dot{x})\dot{x} + c_1 x = 0 \tag{23}$$

in which $f(\dot{x})$ is continuous for all finite $\dot{x}$ and c_1 is a positive constant. The existence of periodic solutions of (22) or (23) is equivalent to the existence of cycles in the phase plane.

In general (22), or (23), has no first integral, but some very general results may be found relating to solutions of these equations, or to those of the equivalent system,

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(y)y - c_1 x. \end{aligned}\right\} \tag{24}$$

For such a system, all the points of the (x,y) phase plane are regular, with the exception of the origin.

If $f(y)$ is identically zero, the curves in the phase plane are concentric ellipses, given by

$$c_1 x^2 + y^2 = \text{constant}. \tag{25}$$

If $f(y) \neq 0$, the slope of the curves in the phase plane is given by

$$\frac{dy}{dx} = -f(y) - c_1\frac{x}{y}. \tag{26}$$

We see that, if $f(y)$ is everywhere positive (except, possibly for $y = 0$), all the curves in the phase plane cross the family of ellipses (25) in the inward direction, and approach the origin as a limit as $t \to \infty$. This is shown for eq. (22) in fig. 4, which $y/\sqrt{c_1}$ is plotted against x for $b_0 = 0$, $\nu\sqrt{c_1} = 1$. We see that the curves spiral round 0; every motion is damped out and no periodic motion exists. Similarly, if $f(y)$ is everywhere negative (except, possibly, for $y = 0$), all trajectories go to infinity as $t \to \infty$.

If $f(y)$ is not of invariable sign, self-sustained oscillations may occur. Thus, if $b_0 < 0$ and $\nu > 0$ (with $c_1 > 0$), on putting $x = \alpha X$, $t = \beta T$, where $\alpha = \sqrt{-b_0/3c_1\nu}$ and $\beta = 1/\sqrt{c_1} > 0$, (22) reduces to Rayleigh's equation

$$\frac{d^2X}{dT^2} + \frac{b_0}{\sqrt{c_1}}\left[\frac{dX}{dT} - \frac{1}{3}\left(\frac{dX}{dT}\right)^3\right] + X = 0, \tag{27}$$

which has a unique limit cycle, which is stable if $b_0 < 0$. This is a closed trajectory surrounding the singular point at 0, as shown for eq. (22) in fig. 5, in which $\sqrt{-\nu/b_0}\, y$ is plotted against $\sqrt{-c_1\nu/b_0}\, x$ for $b_0/\sqrt{c_1} = -1$. Here, $f(y)$ is negative for $\sqrt{-\nu/b_0}|y| < 1$ and positive for $\sqrt{-\nu/b_0}\,|y| > 1$; thus the system has negative damping for $\sqrt{-\nu/b_0}\,|y| < 1$. We note that the maximum value of $\sqrt{-\nu/b_0}\, y$ on the limit cycle is greater than 1 ; thus the limit cycle enters the region of positive damping. The limit cycle divides the phase plane into two regions. Trajectories which start near the origin spiral away from it towards the limit cycle; those which start far from the origin also spiral in towards the same cycle. Thus all motions of the system tend with increasing time to a unique periodic solution.

If c_1 is positive $(= \omega^2)$ and the other parameters of (22) are small, the oscillatory behaviour of the system can be examined by Krylov and Bogoulioubov's method of the first approximation (Minorsky, 1947).

The solution of (22) is written in the form

$$x = A \sin \chi \tag{28}$$

where $\chi = \omega t + \phi$, and A and ϕ are functions of t. If the changes in the amplitude A and the phase ϕ during a cycle are small, it can be shown that the rate of change of A with time is given by

$$\dot{A} = -\tfrac{1}{2} A \left(b_0 + \tfrac{3}{4} \nu A^2 \omega^2\right). \tag{29}$$

When the steady state is reached $\dot{A} = 0$ and thus $A = 0$ or $2\sqrt{-b_0/3c_1\nu}$. Thus, if b_0 and ν are of opposite sign (and small compared with c_1), the system (22) will have a limit cycle in the $(x,\dot{x})$ phase plane of amplitude $2\sqrt{-b_0/3c_1\nu}$. More generally, from (22), it can be shown that the equation of the limit cycle is of the form

$$\sqrt{-\frac{\nu}{b_0}}\, y = f\left(\sqrt{-\frac{c_1\nu}{b_0}}\, x\,,\ \frac{b_0}{\sqrt{c_1}}\right). \tag{30}$$

From figure 5 we see that, even with $b_0/\sqrt{c_1} = 1$, the limit cycle is not very different from a circle of radius $2/\sqrt{3}$ ($= 1.15$), the distortion being much less than that for Van der Pol's equation for the same value of $b_0/\sqrt{c_1}$. Limit cycles for higher values of $b_0/\sqrt{c_1}$ are shown in fig. 6; we see that the intercepts with the y axis do not change appreciably. As $b_0/\sqrt{c_1}$ increases in magnitude, the limit cycle becomes more elongated in the x direction (for a given value of b_0/ν); for larger values of $b_0/\sqrt{c_1}$, the curve approximates to a parallelogram. The variation of the x amplitude with damping is shown in fig. 7; at high values of the damping, $\sqrt{-(c_1\nu/b_0)}\, x_{max}$ increases almost linearly with $(-b_0/\sqrt{c_1})$.

The method of the first approximation shows that, for small values of the damping, the phase ϕ is constant and thus the period is $2\pi/\sqrt{c_1}$. As with the Van der Pol equation, the period increases with the damping. For high values of the damping, the period is approximately $1.6b_0/c_1$.

The periodic solution is stable provided that $b_0 < 0$ and $\nu > 0$.

Case 6 $\nu < 0$, $c_1 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x = 0 \qquad (31)$$

On putting $t = -T$, (31) becomes

$$\frac{d^2x}{dT^2} - b_0\frac{dx}{dT} - \nu\left(\frac{dx}{dT}\right)^3 + c_1 x = 0, \qquad (32)$$

which is of the same form as (22). The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 5. If $b_0 \leq 0$ with $\nu < 0$, all the phase plane trajectories diverge to infinity as $t \to \infty$. If $b_0 > 0$ and $\nu < 0$ (with $c_1 > 0$), there is one limit cycle, which is unstable. The cycle forms the boundary of a domain in the phase plane which includes the origin. Trajectories starting at points within this domain spiral in to the origin (the damping is always positive if $\sqrt{-\nu/b_0}|y| < 1$), whereas all trajectories in the region outside this domain diverge to infinity.

The self-sustained oscillations which occur in cases 5 and 6 can be directly related to those with Van der Pol's equation. Thus on differentiating (22) or (31) with respect to t and putting $z = \dot{x}$ we obtain

$$\ddot{z} + (b_0 + 3\nu z^2)\dot{z} + c_1 z = 0, \qquad (33)$$

which can be reduced to Van der Pol's equation if b_0 and ν are of opposite sign. Equations (22) and (33) occur in certain large amplitude motions of aircraft, such as lateral snaking oscillations (see Briggs and Jones, 1953).

1.4 Systems with damping and negative stiffness

Case 7 $\nu = 0$, $c_1 < 0$.

Equation (9) is then a linear differential equation. This case was dealt with in a previous report (Babister, 1973).

Case 8 $\nu > 0$, $c_1 < 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x = 0 \tag{34}$$

On examining the linear terms in (34) we find that the origin O is the only singular point and is a saddle point (corresponding to an unstable position of equilibrium). All the trajectories tend to infinity as $t \to \infty$, with the exception of the two trajectories (two separatrices of the singular point O) which converge on the origin.

Case 9 $\nu < 0$, $c_1 < 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x = 0 \tag{35}$$

On putting $t = -T$, (35) becomes

$$\frac{d^2x}{dT^2} - b_0\frac{dx}{dT} - \nu\left(\frac{dx}{dT}\right)^3 + c_1 x = 0 \tag{36}$$

which is of the same form as (34). Thus the integral curves are found by applying a time scaling factor $\beta = -1$ to those of case 8. All the trajectories tend to infinity as $t \to \infty$, with the exception of the two arms of the singular point O.

PART 2

Non-linear differential equations of the form

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x + c_3 x^3 = 0$$

2.1 Introduction

We consider solutions of the equation

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x + c_3 x^3 = 0 \tag{37}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y , \\ \dot{y} &= -b_0 y - \nu y^3 - c_1 x - c_3 x^3 , \end{aligned}\right\} \tag{38}$$

where b_0, c_1, c_3 and ν are real constants ($c_3 \neq 0$).

As in part 1, we shall show how the nature of the solution depends upon the initial conditions $x = x_0$, $y = \dot{x}_0 = y_0$ at $t = 0$.

2.2 Systems with cubic-law stiffness ($c_1 = 0$, $c_3 > 0$)

Case 10 $\nu = 0$, $c_1 = 0$, $c_3 > 0$.

This case was considered in a previous report (Babister, 1974). If $b_0 = 0$, all the solutions are periodic, the phase plane trajectories being closed curves (concentric quadrics) surrounding the origin, which is a centre; there are no limit cycles. If $b_0 > 0$, all the trajectories approach the origin as $t \to \infty$; if $b_0 < 0$, all trajectories go to infinity as $t \to \infty$.

Case 11 $\nu > 0$, $c_1 = 0$, $c_3 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_3 x^3 = 0 \tag{39}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y , \\ \dot{y} &= -b_0 y - \nu y^3 - c_3 x^3 \end{aligned}\right\} \tag{40}$$

There is in general no first integral. However, in certain circumstances, a particular integral can be found. Thus, if $\nu b_0^3 = c_3$, (39) has the particular integral

$$\dot{x} = -b_0 x \tag{41}$$

The corresponding trajectories in the (x, y) phase plane are straight lines converging on 0 (since $b_0 > 0$). It can be shown that, if $\nu b_0^3 = c_3$, no trajectories cross the line $y = -b_0 x$. Trajectories to the right of this line (in the first, second and fourth quadrants) turn in towards 0, having a common tangent along Ox at 0. Similarly for trajectories to the left of this line.

Equation (39) is a particular case of the more general equation

$$\ddot{x} + f(\dot{x})\dot{x} + c_3 x^3 = 0 \tag{42}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y , \\ \dot{y} &= -f(y)y - c_3 x^3 . \end{aligned}\right\} \tag{43}$$

The case in which $f(y)$ is identically zero was considered above (case 10), and it was shown that the curves in the phase plane were concentric quadrics. By precisely similar reasoning to that of para. 1.3, it can be shown that, if $f(y)$ is everywhere positive (except, possibly, for $y = 0$), all the curves in the phase plane cross the quadrics of case 10 in the inward direction and approach the origin as a limit as $t \to \infty$. Thus, if in (39) $b_0 \geqslant 0$, every motion is damped out and no periodic motion exists. In particular, no limit cycles occur for the system (40) if b_0 and ν are both positive.

If $f(y)$ is not of invariable sign, periodic solutions and limit cycles may occur, the solution in the phase plane being bounded for $t > t_0$. As in case 5, if $b_0 < 0$ and $\nu > 0$, (39) has a positive damping term if $|y|$ is large and a negative one if $|y|$ is small. On putting $x = \alpha X$, $t = \beta T$ where $\alpha = (-b_0/c_3\nu)^{1/4}$ and $\beta = (-\nu/b_0c_3)^{1/4} > 0$, (39) reduces to

$$\frac{d^2X}{dT^2} + \beta b_0\left[\frac{dX}{dT} - \left(\frac{dX}{dT}\right)^3\right] + X^3 = 0. \tag{44}$$

As shown by Reissig (1956), by means of a topological arguement, (44) has a single limit cycle. This is shown, for eq. (39), in figure 8, in which $\sqrt{-\nu/b_0}\, y$ is plotted against $(-c_3\nu/b_0)^{1/4}\, x$ for $b_0(-\nu/b_0c_3)^{1/4} = -1$. As in case 5, the limit cycle divides the phase plane into two regions. Trajectories which start near the origin spiral away from it, ultimately become asymptotic to the limit cycle; those which start far from the origin also spiral in towards the same cycle. We see that the limit cycle is stable if $b_0 < 0$ and $\nu > 0$.

As $\nu b_0^3/c_3$ increases in magnitude, the limit cycle in the (x,y) plane becomes more elongated in the x direction (as in case 5). However there is relatively little change in the shape of the curve of $\sqrt{-\nu/b_0}\, y$ plotted against $(-c_3\nu/b_0)^{1/4}\, x$, as can be seen from figure 9, which is drawn for $b_0(-\nu/b_0c_3)^{1/4} = -5$. The period P increases with the damping (as shown in fig. 10).

Case 12 $\nu < 0$, $c_1 = 0$, $c_3 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_3x^3 = 0 \tag{45}$$

On putting $t = -T$, (45) becomes

$$\frac{d^2x}{dT^2} - b_0\frac{dx}{dT} - \nu\left(\frac{dx}{dT}\right)^3 + c_3x^3 = 0 \tag{46}$$

Thus the integral curves can be found by applying a scaling factor $\beta = -1$ to those of case 11. If $b_0 \leqslant 0$, the trajectories tend to infinity as t increases. If $b_0 > 0$ and $\nu < 0$, there is one unstable limit cycle. Trajectories which start inside the cycle spiral in towards the origin (which is a stable focus); those which start outside the cycle go off to infinity. Linear trajectories, corresponding to the particular integral (41), occur for $\nu b_0^3 = c_3$.

2.3 <u>Systems with cubic-law stiffness</u> ($c_1 = 0$, $c_3 < 0$)

<u>Case 13</u> $\nu = 0$, $c_1 = 0$, $c_3 < 0$.

<u>Case 14</u> $\nu > 0$, $c_1 = 0$, $c_3 < 0$.

<u>Case 15</u> $\nu < 0$, $c_1 = 0$, $c_3 < 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_3x^3 = 0. \tag{47}$$

Case 13 was considered in a previous report (Babister, 1974). As above the integral curves for case 15 can be found by applying a scaling factor $\beta = -1$ to those of case 14.

There is in general no first integral. However, as in cases 11 and 12 above, a particular integral, given by (41), can be found if $\nu b_0^3 = c_3$. It is readily shown that the only singular point is at the origin, which is a saddle point of the system (47); all the curves in the phase plane tend to infinity as $t \to \pm\infty$, with the exception of the two trajectories (two separatrices of the singular point O) which converge on the origin.

2.4 <u>Systems with cubic stiffness</u> ($c_1 > 0$, $c_3 > 0$)

As pointed out in the general introduction, if both c_1 and c_3 are positive, the origin O is the only singular point. The trajectories in the phase plane are similar to those discussed in paragraph 1.3, and, as we shall see below, limit cycles can occur.

<u>Case 16</u> $\nu = 0$, $c_1 > 0$, $c_3 > 0$.

This case was considered in a previous report (Babister, 1974). If $b_0 = 0$, all the solutions are periodic (O being a centre); there are no limit cycles. If $b_0 > 0$, all trajectories in the phase plane approach the origin as $t \to \infty$; if $b_0 < 0$, all trajectories go to infinity as $t \to \infty$.

<u>Case 17</u> $\nu > 0$, $c_1 > 0$, $c_3 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1x + c_3x^3 = 0 \tag{48}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -b_0 y - \nu y^3 - c_1 x - c_3 x^3 \end{aligned}\right\} \qquad (49)$$

This is the most general case of (37) considered so far. Since c_1 and c_3 both have the same sign, the origin is the only singular point. Eq. (48) has no first integral; this equation is of the form

$$\ddot{x} + f(\dot{x})\dot{x} + g(x) = 0 \qquad (50)$$

with $g(x)$ an odd function of x and $xg(x) > 0$ for all $x \neq 0$. The case $f(\dot{x}) = 0$ has been dealt with in case 16. If $f(\dot{x}) \neq 0$, the slope of the curves in the phase plane is given by

$$\frac{dy}{dx} = -f(y) - \frac{g(x)}{y} \qquad (51)$$

From (51) we see that if $f(y)$ is everywhere positive (except, possibly, for $y = 0$), all the curves in the phase plane cross the simple closed curves of case 16 in the inward direction and approach the origin as a limit as $t \to \infty$.

If $f(y)$ is not of invariable sign, self-sustained oscillations may occur. Thus, if $b_0 < 0$ and $\nu > 0$, as shown by Reissig (1956), the system (49) has a single limit cycle. The closed curve surrounds the singular point O, as shown in fig. 11 (in which $b_0 = -1$, $\nu = 1$, $c_1 = 1$, $c_3 = 1$). If $b_0 < 0$, O is an unstable focus. As in case 5, trajectories which start near the origin spiral away from it towards the limit cycle; those which start far from the origin also spiral in towards the same cycle. The limit cycle is stable if $b_0 < 0$ and $\nu > 0$.

<u>Case 18</u> $\nu < 0$, $c_1 > 0$, $c_3 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x + c_3 x^3 = 0. \qquad (52)$$

On putting $t = -T$, (52) becomes

$$\frac{d^2x}{dT^2} - b_0\frac{dx}{dT} - \nu\left(\frac{dx}{dT}\right)^3 + c_1 x + c_3 x^3 = 0 \qquad (53)$$

which is of the same form as (48). The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 17. If $b_0 \leq 0$, with $\nu < 0$, all the phase plane trajectories diverge to infinity as $t \to \infty$. If $b_0 > 0$ and $\nu < 0$, there is one limit cycle (which is unstable) (as in case 6).

2.5 <u>Systems with cubic stiffness</u> ($c_1 > 0$, $c_3 < 0$)

For these systems there are three singular points, at the origin of 0 and at the points $x = \pm\sqrt{-c_1/c_3}$, these latter points both being saddle points (if $c_1 > 0$). As we shall show below, limit cycles can occur.

<u>Case 19</u> $\nu = 0$, $c_1 > 0$, $c_3 < 0$.

This case was considered in a previous report (Babister, 1974).

If $b_0 = 0$, there is a region surrounding 0 for which cyclic trajectories are possible. If $b_0 > 0$, 0 is a stable focus (or node) and, as shown in that report, <u>some</u> trajectories converge on 0. If $b_0 < 0$, 0 is an unstable focus (or node) and all trajectories go off to infinity as t increases.

<u>Case 20</u> $\nu > 0$, $c_1 > 0$, $c_3 < 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x + c_3 x^3 = 0. \qquad (54)$$

This case has many similarities with that of the previous one (with damping present). Eq. (54) has no general first integral. If $b_0 \geq 0$, the origin 0 is a stable focus and, as shown in fig. 12 (in which $b_0 = 0$, $\nu = 1$, $c_1 = 1$, $c_3 = -1$), trajectories within the region PAQ, P'BQ' converge on 0. If $b_0 < 0$, 0 is an unstable focus. From (54) we see that, in this case, the system has negative damping for small values of $|y|$ and positive damping for large $|y|$, where $y = \dot{x}$. Now, for small values of c_3, the system (54) can be considered as a perturbed Rayleigh's equation, and, by the theory of analytic continuation (Urabe, 1967), the perturbed system will have a unique limit cycle, which is stable if $b_0 < 0$ and $\nu > 0$, and which lies in the neighbourhood of that of Rayleigh's equation.

From the analogue computer results it was found that a stable limit cycle enclosing the origin could occur for $b_0 c_3/\nu c_1^2 < 0.25$, for the range of values investigated. Fig. 13 shows the trajectories and the limit cycle for $b_0 = -1$, $\nu = 1$, $c_1 = 1$ $c_3 = -0.2$. We see that the saddle points $(\pm \sqrt{-c_1/c_3}, 0)$ lie outside the limit cycle, and thus only certain trajectories converge on the limit cycle; the phase plane diagram is comparable with that shown in fig. 12 (in which, however, there is no limit cycle). For larger negative values of c_3, all the trajectories go off to infinity as t increases.

Case 21 $\nu < 0$, $c_1 > 0$, $c_3 < 0$.

The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 20. If $b_0 > 0$, a limit cycle occurs provided that $b_0 c_3/\nu c_1^2 < 0.25$. However, in this case, the limit cycle is unstable, 0 being a stable focus (or node) if $b_0 > 0$.

2.6 Systems with cubic stiffness ($c_1 < 0$, $c_3 > 0$)

For these systems, there are three singular points, at the origin O (a saddle point) and at the points $x = \pm \sqrt{-c_1/c_3}$. As shown below, limit cycles can occur.

Case 22 $\nu = 0$, $c_1 < 0$, $c_3 > 0$.

This case was considered in a previous report (Babister, 1974). If $b_0 = 0$, as shown in fig. 14, there are both outer cyclic trajectories, which enclose the origin and the two centres at $\pm \sqrt{-c_1/c_3}$, and also inner cyclic trajectories which enclose one of the centres (together with the figure 8 curve OPOQO which is a separatrix for the system).

If $b_0 > 0$, trajectories tend to stable foci at $x = \pm\sqrt{-c_1/c_3}$ as $t \to \infty$; if $b_0 < 0$, trajectories go to infinity as $t \to \infty$, the foci at $\pm\sqrt{-c_1/c_3}$ being unstable.

Case 23 $\nu > 0$, $c_1 < 0$, $c_3 > 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x + c_3 x^3 = 0 \tag{55}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -b_0 y - \nu y^3 - c_1 x - c_3 x^3 . \end{aligned}\right\} \tag{56}$$

There is in general no first integral. However, in certain circumstances, a particular integral can be found. Thus (55) has the particular integral

$$\dot{x} = \lambda x \tag{57}$$

where

$$\nu\lambda^3 + c_3 = 0, \tag{58}$$

λ being a real root of

$$\lambda^2 + b_0\lambda + c_1 = 0. \tag{59}$$

Eliminating λ from (58) and (59) we find that linear trajectories occur in the (x,y) phase plane if

$$c_3^2 - b_0^3\nu c_3 + 3b_0\nu c_1 c_3 + \nu^2 c_1^3 = 0 \tag{60}$$

with $b_0^2 \geq 4c_1$, there being only one such trajectory for any set of values of b_0, ν, c_1, c_3 satisfying (58) and (59). These trajectories converge on O (since $\nu/c_3 > 0$).

More generally, if $b_0 \geq 0$, the trajectories converge on the stable foci (or nodes) at the points $x = \pm\sqrt{-c_1/c_3}$ (apart from the two trajectories which terminate at the saddle point O). If $b_0 < 0$, the foci (or nodes) are unstable. From (55) we see that, in this case, the system has negative damping for small values of $|y|$ and positive damping for large $|y|$. Now, for small values of c_1, the system (55) can be considered as a perturbed form of eq. (39). By the theory of analytic continuation (as for case 20), we find that, if $c_3 > 0$, the system (56) with $b_0 < 0$ will have a stable limit cycle enclosing the origin for small values of c_1. In fact, a limit cycle occurs for a wide range of negative values of c_1; fig. 15 is drawn for $b_0 = -1$, $\nu = 1$, $c_1 = -1$, $c_3 = 1$. The general shape of the limit cycle in this case is very similar to that of the outermost curve in fig. 14. Both curves have practically the same maximum amplitudes in both x and y, the maximum y amplitude being greater than 1. It is readily shown, by considering the slopes of the limit cycle and of the cyclic curves in fig. 14, that the limit cycle crosses any cyclic curve in the outward direction (away from 0) for $|y| < \sqrt{-b_0/\nu}$, and in the inward direction for $|y| > \sqrt{-b_0/\nu}$. An upper bound to the values of $(-c_1)$ for which this type of limit cycle can occur is found by noting that, as shown above, (56) has linear trajectories passing through the origin if eq. (60) is satisfied. This is the curve FOG shown in fig. 16, in which the coordinates are c_1/b_0^2 and $(-c_3/\nu b_0^3)$. It can be shown, from (56), that if x has the dimensions of length, these coordinates are non-dimensional, and the equation of any closed trajectory can be written in the form

$$\sqrt{-\frac{\nu}{b_0}}\, y = f\left(\sqrt{-\frac{c_1 \nu}{b_0}}\, x, \frac{c_1}{b_0^2}, \frac{c_3}{\nu b_0^3}\right) \tag{61}$$

From the analogue computer results it was found that limit cycles could occur for all positive values of c_3, as indicated in fig. 16. For the system (56) with $b_0 < 0$, two limit cycles can occur for values of c_1 and c_3 in the region GOH, these stable cycles each surrounding an unstable focus (at $x = \pm\sqrt{-c_1/c_3}$). Figures 17 and 18 show the trajectories and limit cycles for $b_0 = -1$, $\nu = 1$, $c_1 = -1$, $c_3 = 0.3$ and 0.2, the latter point being within the region GOH. In both cases

O is a saddle point, this point being external to the two limit cycles shown in fig. 18. We note that the maximum y amplitude of these limit cycles is only slightly greater than unity.

The critical values of the parameters for the occurrence of the different types of limit cycle are also shown in fig. 19, in which $\mu = \sqrt{|c_3 b_0/c_1^2 \nu|}$ is plotted against $|c_1|/b_0^2$. For the range of parameters shown in this diagram, the curves corresponding to OC and OG coincide (within the margin of experimental error), μ being approximately equal to $\frac{1}{2}$ for $1 < |c_1|/b_0^2 < 4$. However $\mu \to 0$ as $|c_1|/b_0^2 \to 0$. For points on OG, from (60), $c_3 \sim \nu(-c_1)^{3/2}$ for large negative values of c_1, and thus $\mu \sim (b_0^2/ |c_1|)^{1/4}$.

<u>Case 24</u> $\nu < 0$, $c_1 < 0$, $c_3 > 0$.

The integral curves for this case are found by applying a time scaling factor $\beta = -1$ to those of case 23. Linear trajectories occur in the (x,y) plane if (60) is satisfied with $b_0^2 \geqslant 4c_1$. If $b_0 > 0$, one or two limit cycles occur, depending on whether the corresponding point is outside the region GOH (fig. 16) or within it. However, in this case, the limit cycles are unstable.

2.7 <u>Systems with cubic stiffness</u> ($c_1 < 0$, $c_3 < 0$)

For these systems, the origin O is the only singular point, and is a saddle point.

<u>Case 25</u> $\nu = 0$, $c_1 < 0$, $c_3 < 0$.

<u>Case 26</u> $\nu > 0$, $c_1 < 0$, $c_3 < 0$.

<u>Case 27</u> $\nu < 0$, $c_1 < 0$, $c_3 < 0$.

$$\ddot{x} + b_0\dot{x} + \nu\dot{x}^3 + c_1 x + c_3 x^3 = 0 \tag{62}$$

or the equivalent system

$$\left.\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -b_0 y - \nu y^3 - c_1 x - c_3 x^3 \end{aligned}\right\} \quad (63)$$

There is in general no first integral (if $b_0 \neq 0$). However, as in cases 23 and 24, (62) has a particular integral of the form

$$\dot{x} = \lambda x \quad (64)$$

if (60) is satisfied, with $b_0^2 \geqslant 4c_1$.

Case 25 was considered in a previous report (Babister, 1974). The integral curves for case 27 can be found by applying a scaling factor $\beta = -1$ to those of case 26. In all three cases it can readily be shown that all the curves in the phase plane tend to infinity as $t \to \infty$, with the exception of the two trajectories (two separatrices of the singular point O) which converge on the origin.

References

Babister, A.W. Non-linear differential equations having quadratic stiffness terms. University of Glasgow, Department of Aeronautics and Fluid Mechanics. Report 7301 (1973).

Babister A.W. Non-linear differential equations having cubic stiffness terms. University of Glasgow, Department of Aeronautics and Fluid Mechanics. Report 7401 (1974).

Babister, A.W. Non-linear differential equations having both quadratic damping and stiffness. University of Glasgow, Department of Aeronautics and Fluid Mechanics. Report 7501 (1975).

Briggs, B.R. and Jones, A.L. Techniques for calculating parameters of non-linear dynamic systems from response data. N.A.C.A. Tech. Note 2977 (1953).

Loud W.S. Behaviour of the period of solutions of certain plane autonomous systems near centres. Contributions to Differential equations. (Interscience, 1964).

Minorsky, N. Introduction to non-linear mechanics. (J.W. Edwards, Ann Arbor, 1947).

Reissig, R. Selbsterregung eines einfachen Schwingers.
Math. Nachr., 15 (1956), 191 - 196.

Urabe, M. Non-linear autonomous oscillations.
(Academic Press, 1967).

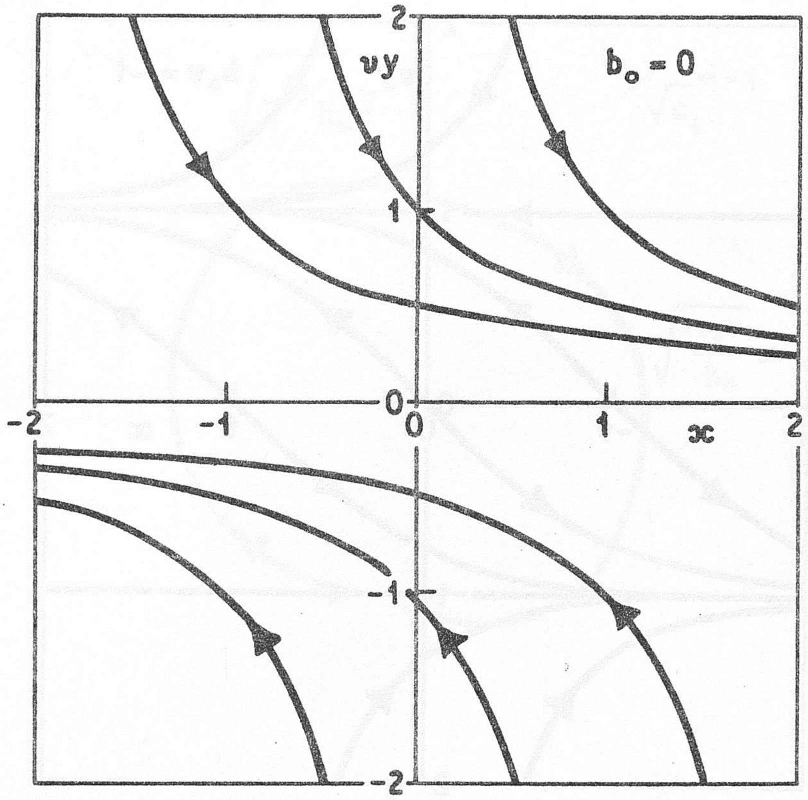

Fig. 1. TRAJECTORIES $\upsilon > 0$, $C_1 = 0$, $C_3 = 0$

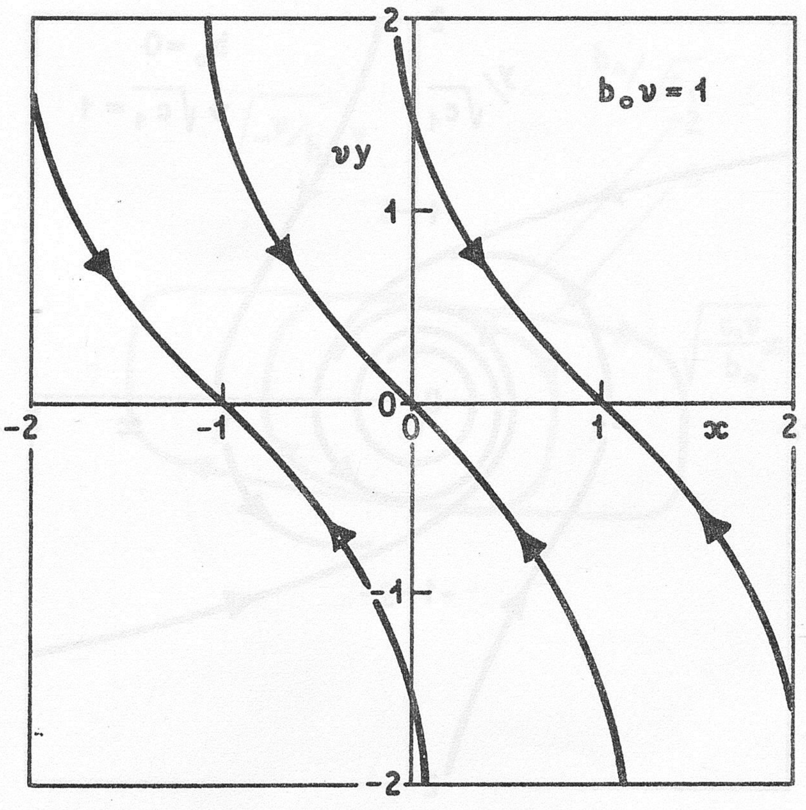

Fig. 2. TRAJECTORIES $\upsilon > 0$, $C_1 = 0$, $C_3 = 0$

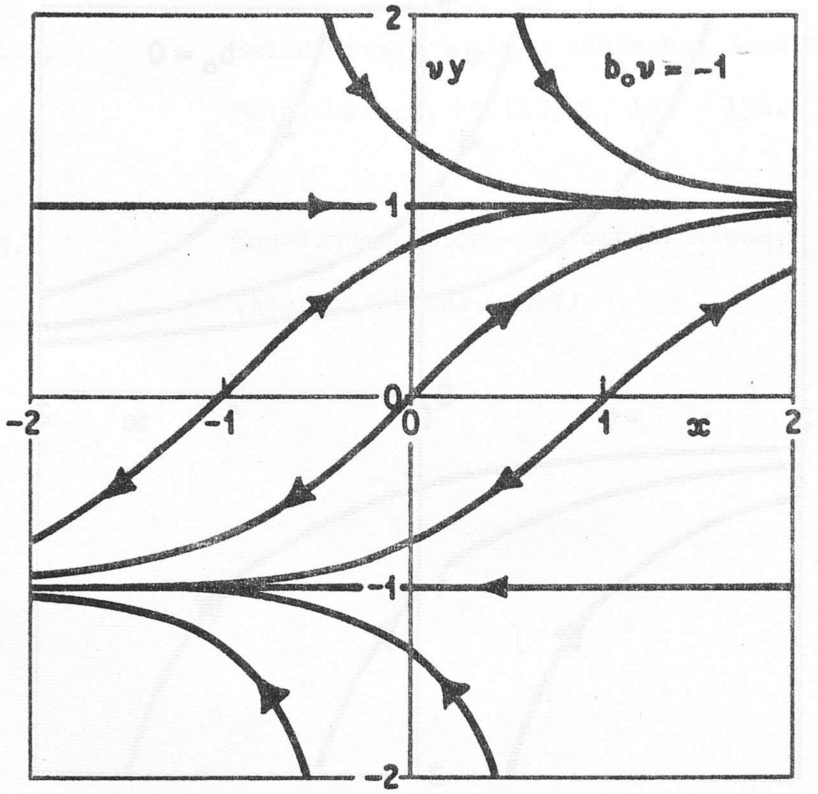

Fig. 3. TRAJECTORIES $\nu > 0,\ C_1 = 0,\ C_3 = 0$

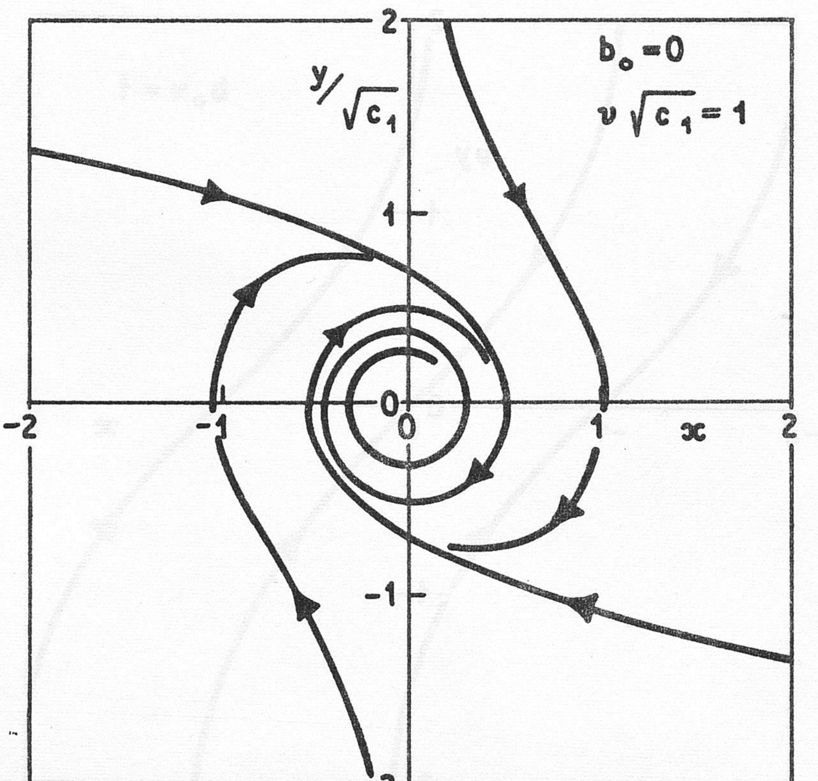

Fig. 4. TRAJECTORIES $\nu > 0,\ C_1 > 0,\ C_3 = 0$

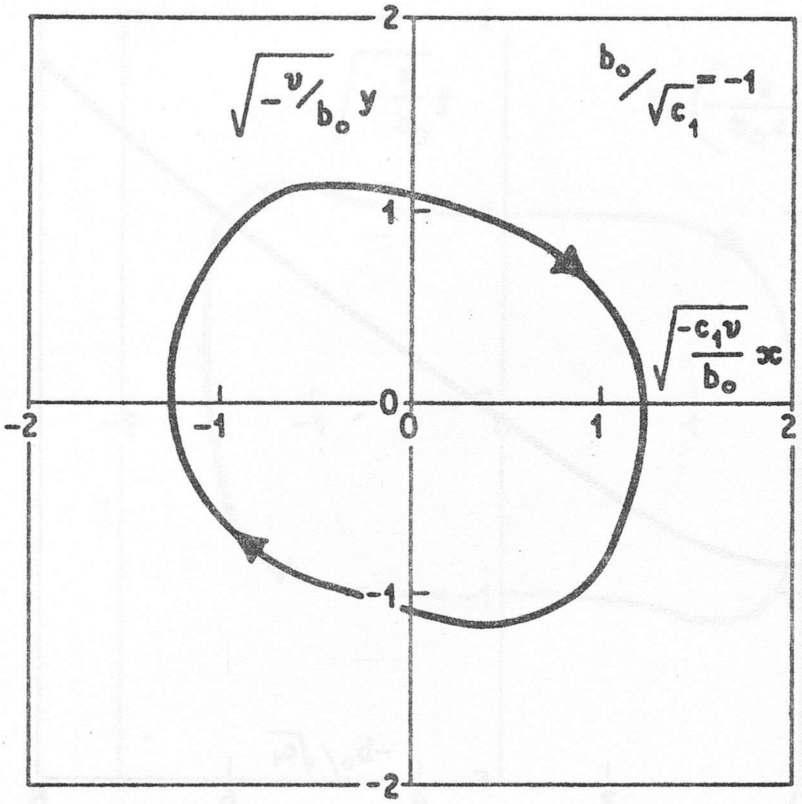

Fig.5. LIMIT CYCLE $\nu > 0,\ C_1 > 0,\ C_3 = 0$

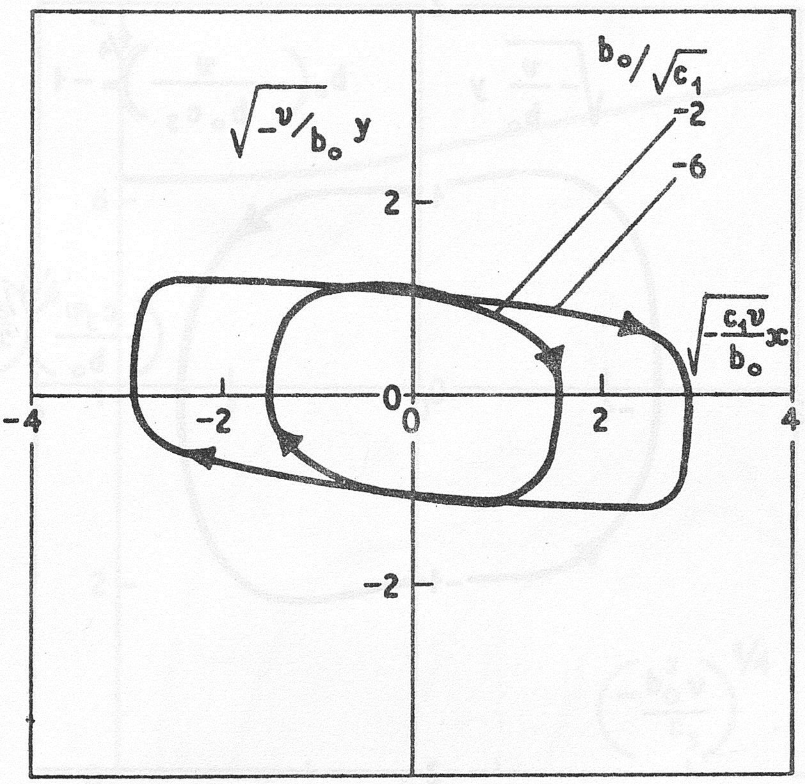

Fig.6. LIMIT CYCLE $\nu > 0,\ C_1 > 0,\ C_3 = 0$

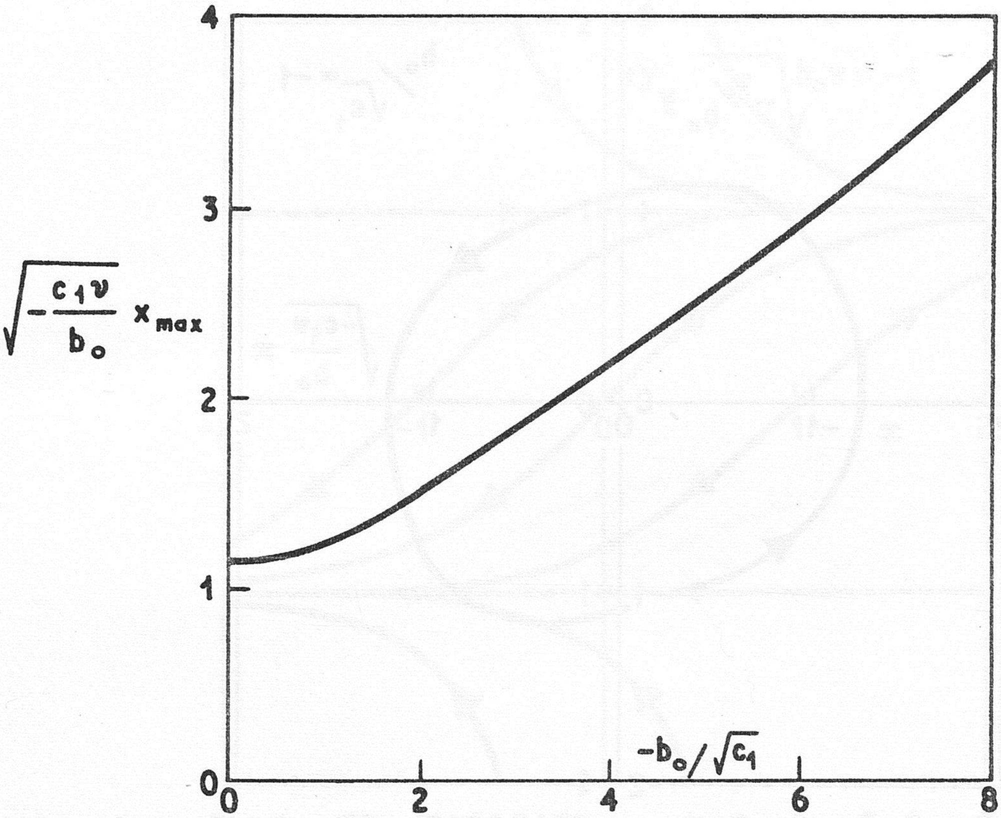

Fig.7. x AMPLITUDE OF LIMIT CYCLE

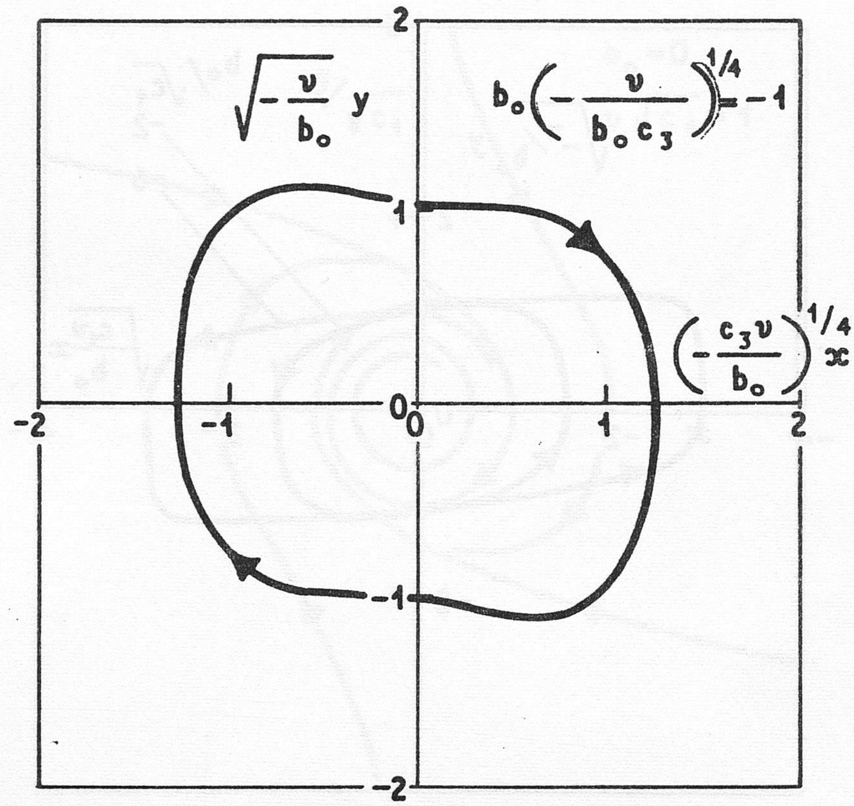

Fig. 8. LIMIT CYCLE $v > 0$, $C_1 = 0$, $C_3 > 0$

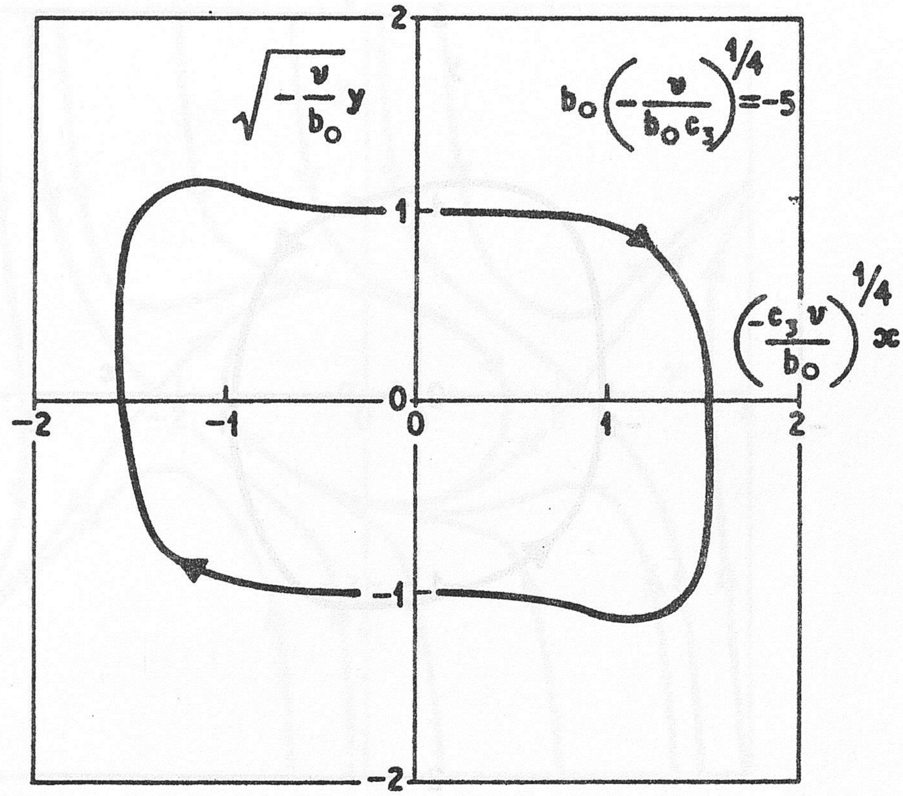

Fig. 9. LIMIT CYCLE $v > 0$, $C_1 = 0$, $C_3 > 0$

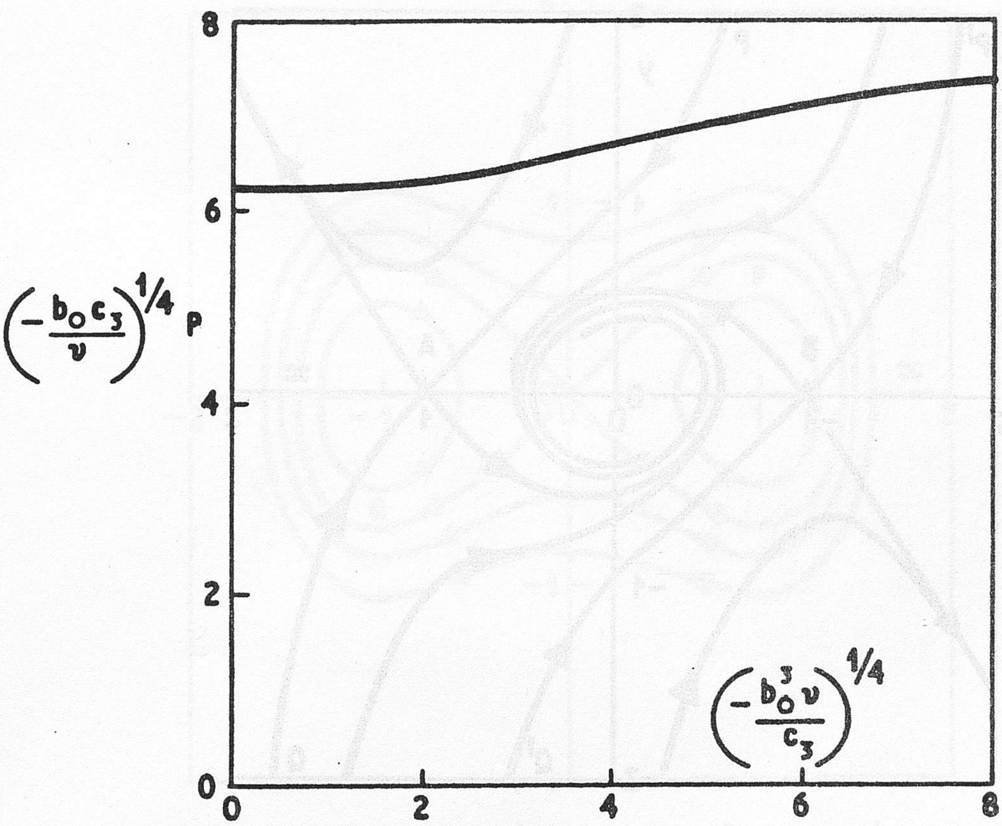

Fig. 10. PERIOD P OF LIMIT CYCLE

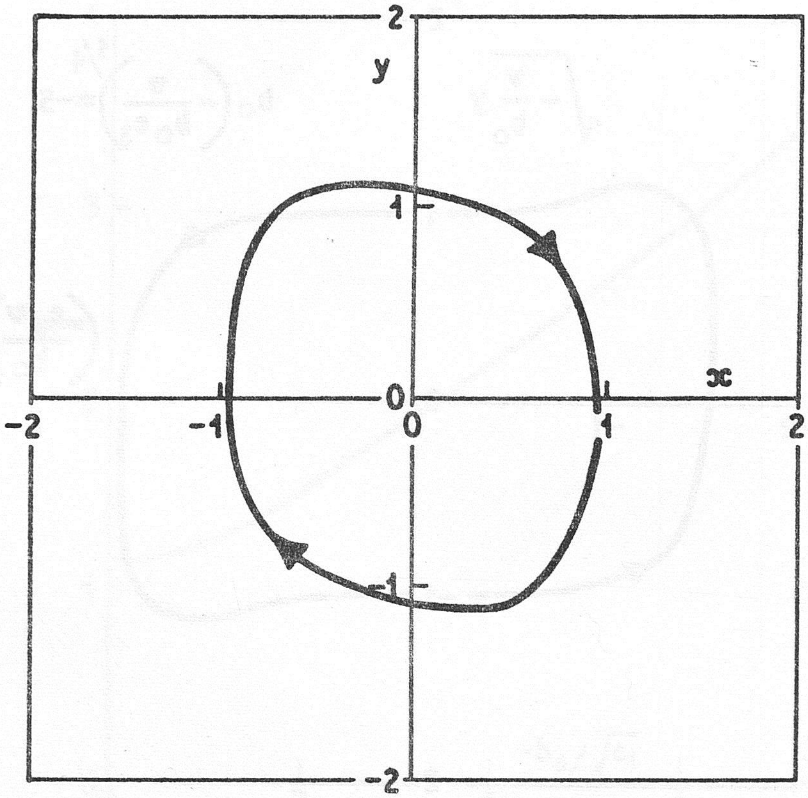

Fig. 11. LIMIT CYCLE $b_o = -1$, $\nu = 1$, $C_1 = 1$, $C_3 = 1$

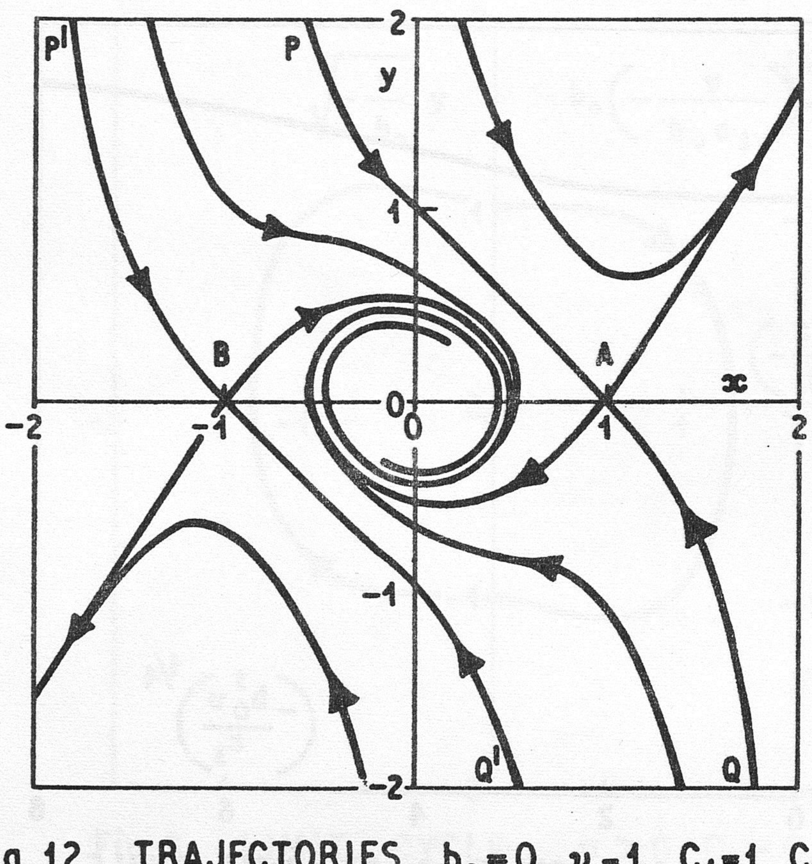

Fig. 12. TRAJECTORIES $b_o = 0$, $\nu = 1$, $C_1 = 1$, $C_3 = -1$

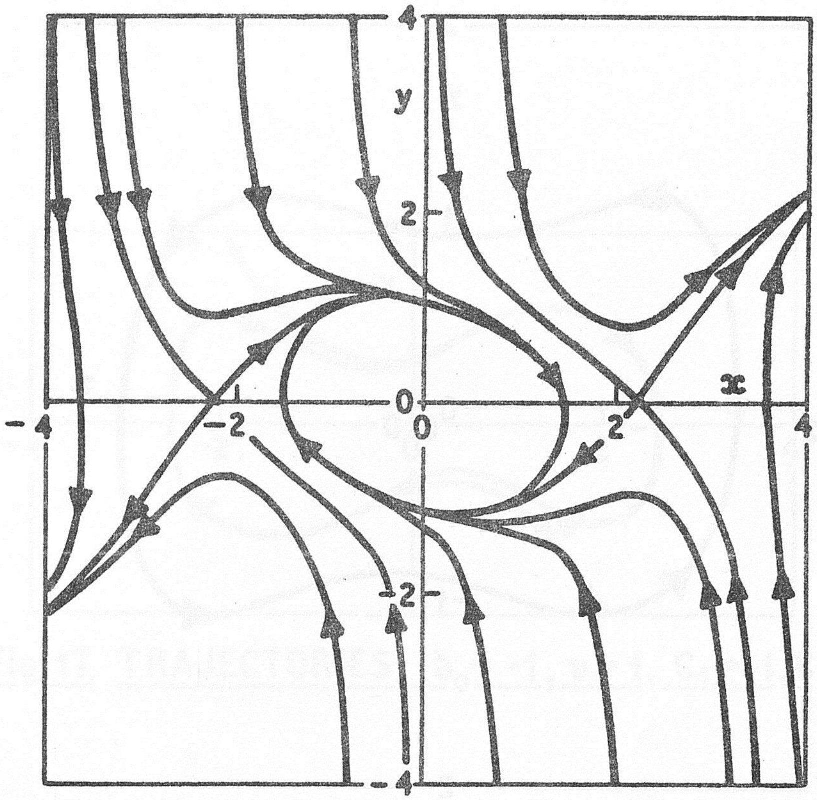

Fig. 13. TRAJECTORIES $b_0 = -1$, $\nu = 1$, $c_1 = 1$, $c_3 = -0\cdot2$

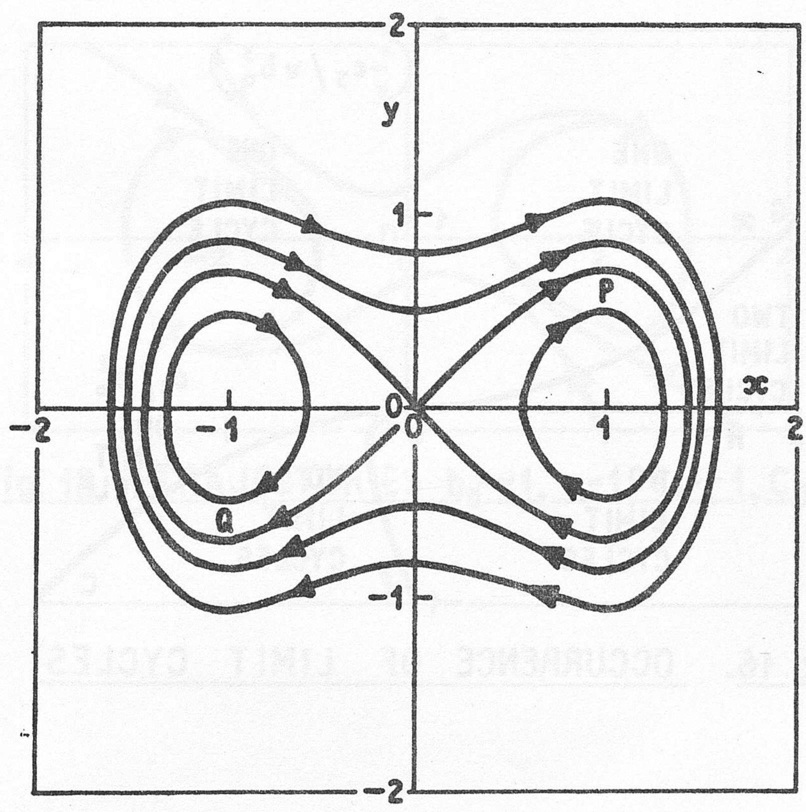

Fig. 14. TRAJECTORIES $b_0 = 0$, $\nu = 0$, $c_1 = -1$, $c_3 = 1$

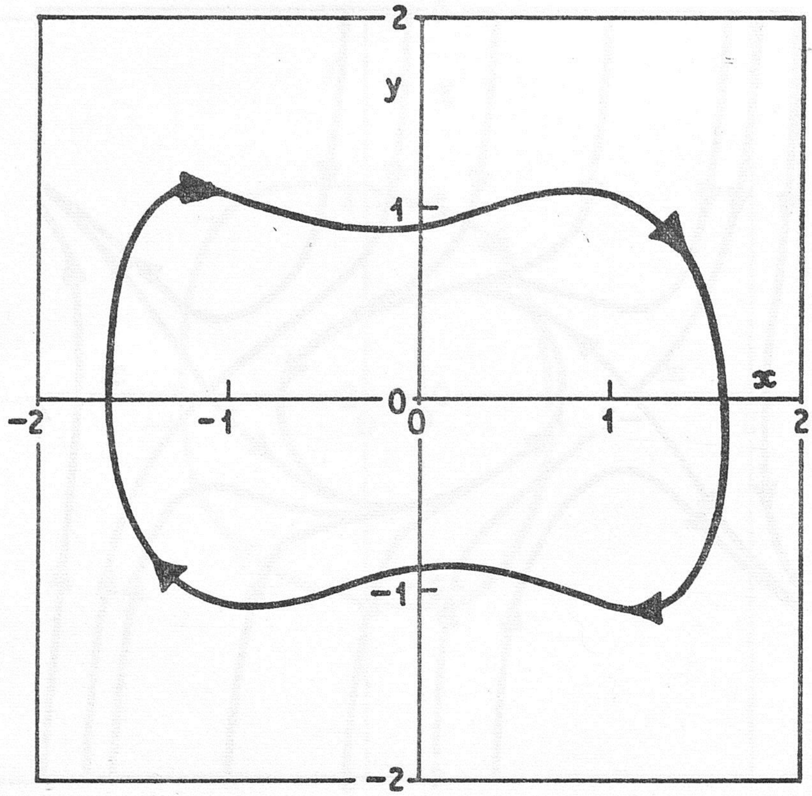

Fig. 15. LIMIT CYCLE $b_0 = -1$, $\nu = 1$, $C_1 = -1$, $C_3 = 1$

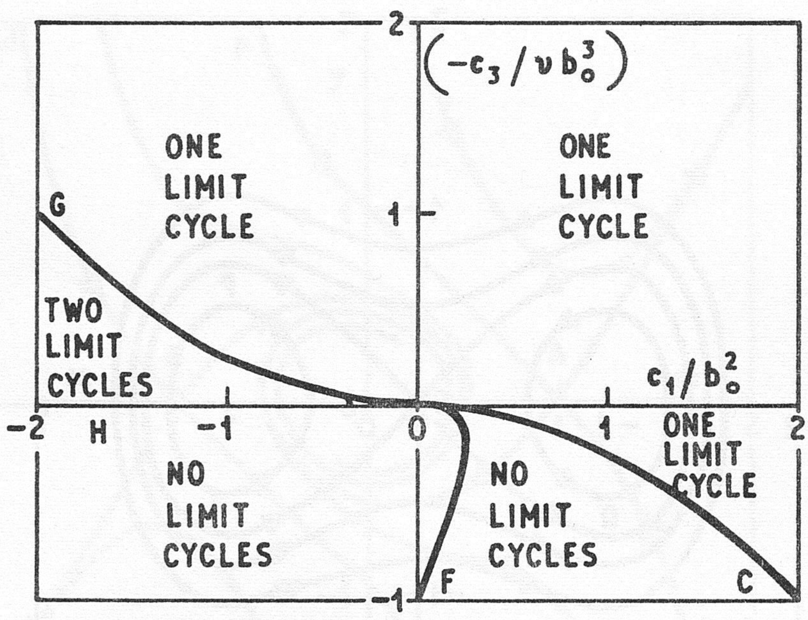

Fig. 16. OCCURRENCE OF LIMIT CYCLES

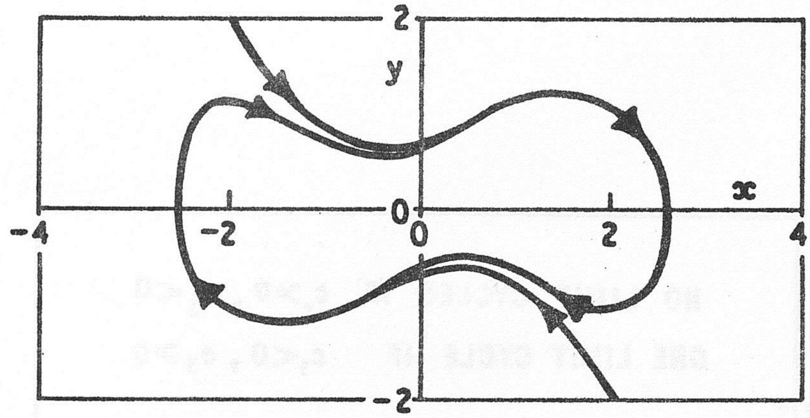

Fig. 17. TRAJECTORIES $b_0 = -1$, $\upsilon = 1$, $C_1 = -1$, $C_3 = 0{\cdot}3$

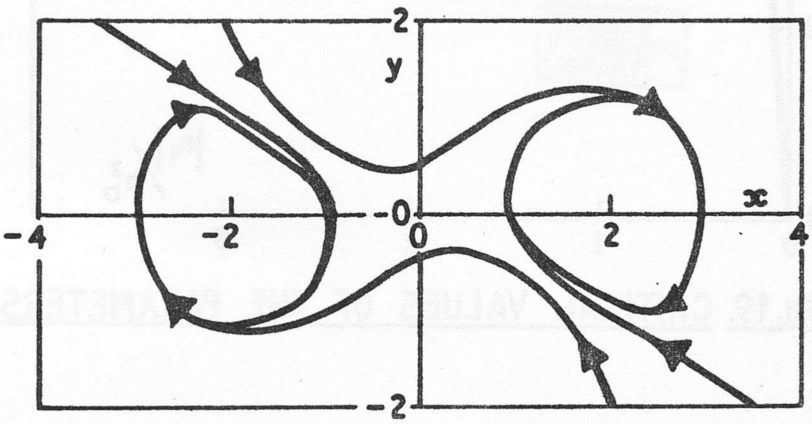

Fig. 18. TRAJECTORIES $b_0 = -1$, $\upsilon = 1$, $C_1 = -1$, $C_3 = 0{\cdot}2$

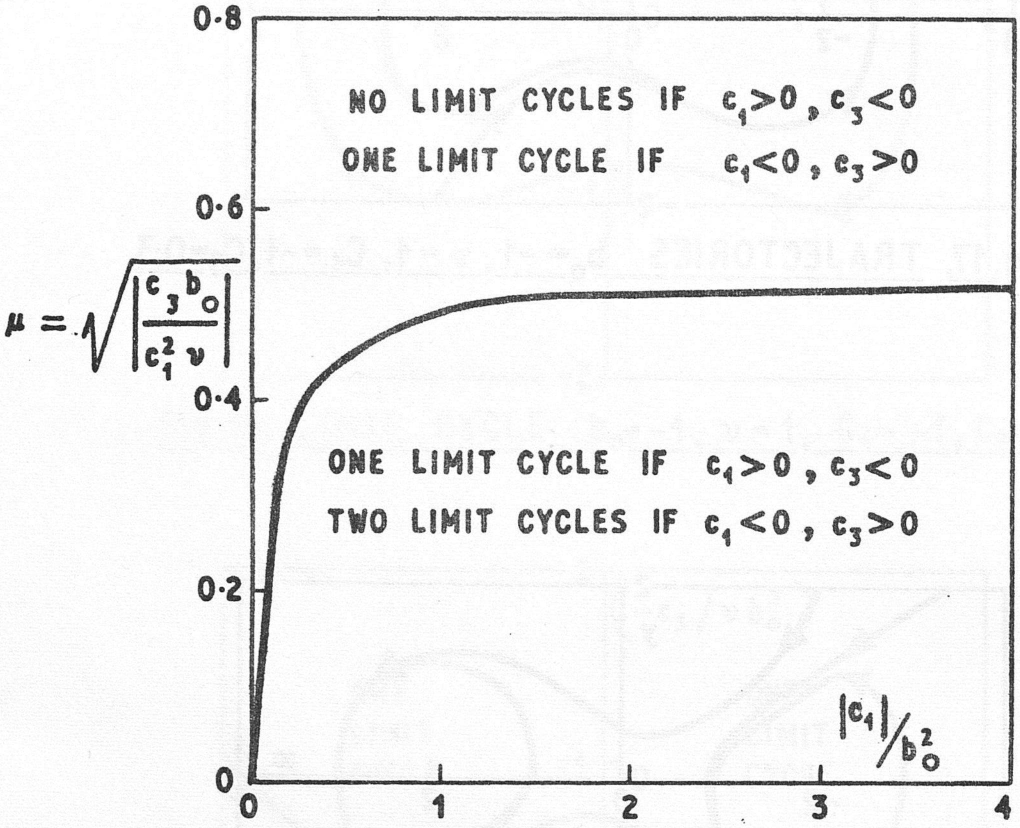

Fig.19. CRITICAL VALUES OF THE PARAMETERS